

INDECOMPOSABLE AND NONCROSSED PRODUCT DIVISION ALGEBRAS OVER FUNCTION FIELDS OF SMOOTH p -ADIC CURVES

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ABSTRACT. We construct indecomposable and noncrossed product division algebras over function fields of smooth curves X over \mathbb{Z}_p . This is done by defining an index preserving morphism $s : \text{Br}(\hat{K}(X))' \rightarrow \text{Br}(K(X))'$ which splits $\text{res} : \text{Br}(K(X)) \rightarrow \text{Br}(\hat{K}(X))$, where $\hat{K}(X)$ is the completion of $K(X)$ at the special fiber, and using it to lift indecomposable and noncrossed product division algebras over $\hat{K}(X)$.

1. INTRODUCTION

Let X be a smooth projective curve over $S = \text{Spec } \mathbb{Z}_p$, let $F = K(X)$ be its function field, and let $\hat{K}(X)$ denote the completion of $K(X)$ with respect to the discrete valuation on $K(X)$ defined by the special fiber X_0 . We define an index-preserving homomorphism

$$\text{Br}(\hat{K}(X))' \rightarrow \text{Br}(K(X))'$$

that splits the restriction map $\text{res} : \text{Br}(K(X))' \rightarrow \text{Br}(\hat{K}(X))'$. Here the “prime” denotes “prime-to- p ”. The field $\hat{K}(X)$ is not unlike a power series field over a number field, and using the methods of [Bru95] and [Bru96], we construct certain exotic kinds of division algebras over $\hat{K}(X)$, and transfer these constructions to $K(X)$ using our homomorphism. In particular, we have a new construction of noncrossed product division algebras and indecomposable division algebras of unequal period and index over the rational function field $\mathbb{Q}_p(t)$.

Recall if K is a field, a K -division algebra D is a division ring that is finite-dimensional and central over K . The *period* of D is the order of the class $[D]$ in $\text{Br}(K)$, and the *index* $\text{ind}(D)$ is the square root of D ’s K -dimension. A *noncrossed product* is a K -division algebra whose structure is not given by a Galois 2-cocycle. Noncrossed products were first constructed by Amitsur in [Ami72], settling a longstanding open problem. Since then there have been several other constructions, including [Sal78], [JW90], [Bru95], [Bru01], [RY01], and [Han04]. Saltman recently showed that all division algebras of prime degree over our fields are cyclic ([Sal07]); the indexes of our examples are all divisible by the square of a prime.

A K -division algebra is *indecomposable* if it cannot be expressed as the tensor product of two nontrivial K -division algebras. It is easy to see that all

division algebras of equal period and index are indecomposable, and that all division algebras of composite period are decomposable, so the problem of producing an indecomposable division algebra is only interesting when the period and index are unequal prime-powers. Albert constructed decomposable division algebras of unequal (2-power) period and index in the 1930's, but indecomposable division algebras of unequal period and index did not appear until [Sal79] and [ART79]. Since then there have been several constructions, including [Tig87], [JW90], [Jac91], [SVdB92], [Kar98], [Bru96], and [McK08].

Noncrossed products over a rational function field $K(t)$ were constructed in [Bru01], for any p -adic field K . However the construction here is much more general, and our fields constitute a much larger class. For example, our methods apply to fields such as $K(X) = \mathbb{Q}_p(t)(\sqrt{t^3 + at + b})$, where $a, b \in \mathbb{Z}_p$, and $p \neq 2, 3$ does not divide the discriminant $4a^3 + 27b^2$. For here $K(X)$ is the function field of the elliptic curve $X = \text{Proj } \mathbb{Z}_p[x, y, z]/(y^2z - x^3 - axz^2 - bz^3)$ (with $t = x/z$), which is smooth by [Liu02], IV.3.30 and IV.3.35. Nevertheless, it is well known that not all finite extensions of $\mathbb{Q}_p(t)$ are function fields of smooth curves over \mathbb{Z}_p , as we will indicate, and we have no construction for these.

Notation. Throughout this paper we let (c) denote the image of $c \in K^*$ in $H^1(K, \mu_n)$. In general we write $a.b$ for the cup product of cohomology classes a and b , unless $a \in H^1(K, \mathbb{Q}/\mathbb{Z})$ and $b = (c)$, in which case for historical reasons we write

$$(a, c) = a.(c) \in \text{Br}(K)$$

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2. TAMELY RAMIFIED COVERS OF SMOOTH CURVES

In this section we review some facts about smooth curves over complete discrete valuation rings and tamely ramified covers of them. We follow the terminology of [KM85] and [Wew99].

2.1. Smooth Curves and Marks. Let R be a noetherian ring. By a *smooth curve X over R* we mean a scheme X which is projective and smooth of relative dimension 1 over $\text{Spec } R$. In particular, X is flat and of finite presentation over $\text{Spec } R$.

By a *mark D on X* we mean an effective étale relative Cartier divisor D on X , that is, D is a closed subscheme of X whose defining ideal is an invertible \mathcal{O}_X -module and such that D is étale over $\text{Spec } R$.

Note that the definitions of smooth curves, effective relative Cartier divisors, and marks are stable under arbitrary base change (see [Gro67] 17.3.3 (iii), [Gro61] 5.5.5 (iii), and [KM85] 1.1.4).

In this paper we work with smooth curves over complete discrete valuation rings. In the next lemma we collect some useful facts about them.

Lemma 2.1. *Let (R, \mathfrak{m}, k) be a complete noetherian local domain with maximal ideal \mathfrak{m} , residue field $k = R/\mathfrak{m}$, and field of fractions $K = \text{Frac } R$. Let X be a smooth curve over R and write $X_0 \stackrel{\text{df}}{=} X \times_{\text{Spec } R} \text{Spec } k$ for its special fiber (a smooth curve over k). For any effective relative Cartier divisor D on X , denote its restriction to X_0 by $D_0 \stackrel{\text{df}}{=} D \times_{\text{Spec } R} \text{Spec } k$.*

- (i) *If R is regular then both X and X_0 are regular.*
- (ii) *If X is connected then X_0 is connected.*
- (iii) *Any effective relative Cartier divisor D on X is finite over $\text{Spec } R$. In particular, we may write $D = \text{Spec } S$ where S is a product of finite free local R -algebras.*
- (iv) *An effective relative Cartier divisor D on X is a mark if and only if D_0 is a mark on X_0 .*
- (v) *Assume that R is normal. Then a mark D is irreducible or integral if and only if it is connected. Hence there is a 1-1 correspondence between irreducible components of a mark D and those of D_0 , and in particular, if D is an integral mark, then so is D_0 .*
- (vi) *If D is an integral mark then $[K(D) : K] = [k(D_0) : k]$ where $K(D)$ and $k(D_0)$ denote the function fields of D and D_0 respectively.*
- (vii) *If R is a discrete valuation ring then any irreducible effective Cartier divisor on X other than the irreducible components of X_0 is relative. Moreover any mark D_0 on X_0 lifts to a mark D on X .*

Proof. Since X and X_0 are smooth over $\text{Spec } R$ and $\text{Spec } k$ respectively, (i) follows from [Gro67] 17.5.8 (iii). On the other hand (ii) is just a special case of [Gro67] 18.5.19.

Since $D \rightarrow \text{Spec } R$ is proper (it is the composition of the closed immersion $D \hookrightarrow X$ and the projective morphism $X \rightarrow \text{Spec } R$), the first assertion of (iii) follows from [KM85] 1.2.3. The second assertion follows from the fact that (by definition) finite morphisms are affine, that any finite algebra S over a henselian ring R is a product of finite local R -algebras (see [Mil80] I.4.2 (b)), and that a finitely generated module over a local ring is flat if and only if it is free (see [Mat89] 7.10). This proves (iii).

To prove (iv) we may assume by (iii) that $D = \text{Spec } S$ for some finite free (hence flat) local R -algebra S , and it remains to show that S is unramified over R if and only if $S \otimes_R k$ is unramified over k . This follows from [Gro67] 17.4.1 (a),(d) since S , being a local ring, is unramified over R if and only if it is unramified over R at its maximal ideal.

To prove (v), first observe that if D is a mark, then it is reduced by [Gro63] I.9.2 since R is a domain. Hence a mark is irreducible if and only if it is integral. Clearly if D is irreducible then it must be connected; conversely, since $D \rightarrow \text{Spec } R$ is étale and R is normal, D is also normal ([Gro63] I.9.10), hence if D is connected it must be irreducible. Therefore connected and irreducible components of D agree, and since $D \rightarrow \text{Spec } R$ is proper and R is henselian the rest of (v) follows directly from [Gro67] 18.5.19 (or [Mil80] I.4.2).

To prove (vi), write $D = \operatorname{Spec} S$ for some finite free local R -algebra S using (iii). Note that $[S \otimes_R K : K] = [S \otimes_R k : k]$ equals the rank of S over R , hence it is enough to show that $S \otimes_R K = K(D)$ and $S \otimes_R k = k(D_0)$. Since S is étale over R , $\mathfrak{m}S$ is the maximal ideal of S and $S/\mathfrak{m}S = S \otimes_R k = k(D_0)$ is its residue field; on the other hand, $S \otimes_R K$ is a localization of S that contains S and is étale over K , hence we must have $S \otimes_R K = \operatorname{Frac} S = K(D)$.

Finally the first fact in (vii) follows from [Liu02] IV.3.10. The second assertion is then a consequence of (iv) and [Gro67] 21.9.11 (i) (see also the proof of 21.9.12). \square

2.2. Tamely ramified covers. Let K be a field and $v: K \rightarrow \mathbb{Z} \cup \{\infty\}$ be a discrete valuation with residue field of characteristic p . Let L/K be a finite separable field extension and L' be the Galois closure of L (in some separable closure of K containing L). Let $\{w_i\}$ be the discrete valuations of L' extending v and denote by I_i their inertia groups (see [Bou98] V.2.3 or [Lan02] VII.2). Recall that L/K is said to be *tamely ramified* with respect to v if p does not divide $|I_i|$ for all i .

Let X be an integral smooth curve over a complete regular noetherian local ring (R, \mathfrak{m}, k) . By Lemma 2.1 (i) X is regular (and thus normal) so that each irreducible effective Weil (or Cartier) divisor E defines a discrete valuation on the function field $K(X)$ of X , which we will denote by v_E . Now let D be a mark on X and $\rho: Y \rightarrow X$ be a finite $(\operatorname{Spec} R)$ -morphism of integral smooth curves over R . We say that ρ is a *tamely ramified cover* of the pair (X, D) if it is étale over $X - D$ and tamely ramified along D , that is, the function field $K(Y)$ of Y is a tamely ramified extension of the function field $K(X)$ of X with respect to the valuations defined by irreducible components of D . Étale locally, tamely ramified covers have the following description (see [Wew99] 2.3.4 and [FK88] A.I.11): for each geometric closed point $y: \operatorname{Spec} \Omega \rightarrow Y$ with image $x = \rho \circ y: \operatorname{Spec} \Omega \rightarrow X$ there exist affine étale neighborhoods $\operatorname{Spec} B \rightarrow Y$ and $\operatorname{Spec} A \rightarrow X$ of y and x such that $B = A[w]/(w^n - z)$ for some $z \in A$ (an étale local coordinate of D) and some integer n prime to the characteristic of k .

Lemma 2.2. *Let X be an integral smooth curve over a complete regular noetherian local ring (R, \mathfrak{m}, k) , and D be a mark on X . Let $\rho: Y \rightarrow X$ be a tamely ramified cover of (X, D) . Then*

- (i) *Y is flat over X and equals the normalization of X in $K(Y)$;*
- (ii) *any integral relative effective Cartier divisor F on Y lying over D or over a mark E on X such that $E \cap D = \emptyset$ is itself a mark;*
- (iii) *let F be an irreducible mark on Y lying over an irreducible mark E on X . Then then the ramification (resp. the inertia) degree of v_F over v_E equals the ramification (resp. inertia) degree of v_{F_0} over v_{E_0} .*

Proof. The restriction $\rho_0: Y_0 \rightarrow X_0$ of ρ to the special fibers is a finite generically étale map between smooth curves over a field, which is flat by

[Gro63] IV.1.3 (ii) for instance. Hence ρ is also flat by the local criterion of flatness (see [Gro63] IV.5.9). To finish the proof of (i), note that Y is regular (Lemma 2.1 (i)) and thus normal, and since it is also integral over X , it equals the normalization of X in $K(Y)$.

To prove (ii), assume first that F lies over a mark E on X with $E \cap D \neq \emptyset$. Since $\rho^{-1}(X - D) \rightarrow X - D$ is étale by assumption, so is $\rho^{-1}E \rightarrow E$ by base change; but E is a mark, hence the composition $\rho^{-1}E \rightarrow E \rightarrow \operatorname{Spec} R$ is also étale, showing that $\rho^{-1}E$ is a mark. On the other hand, F is an irreducible component of $\rho^{-1}E$, hence also a connected component by Lemma 2.1 (v), showing that F is a mark as well.

Now suppose that F lies over D . We may assume that D is connected, and hence by Lemma 2.1 (v) that D is an affine integral local scheme, which is regular since R is regular and $D \rightarrow \operatorname{Spec} R$ is étale (see [Mil80] I.3.17 (c)). To show that F is a mark, we need to check that $F \rightarrow D$ is étale, which can be done after the faithfully flat base extension $\operatorname{Spec} A \rightarrow X$, where $A = \mathcal{O}_{X, \overline{x_0}}$ is the strict henselization at the closed point x_0 of D , by [Gro67] 17.7.3 (ii). Thus we may assume that $X = \operatorname{Spec} A$ is regular and strictly local ([Gro67] 18.8.13 (c)), that $D = \operatorname{Spec} A/(z)$ where z is part of a regular system of parameters (ibid. since $A/(z)$ is the strict henselization of $\Gamma(D, \mathcal{O}_D)$), and that $Y = \operatorname{Spec} B$ where B is a finite product of rings of the form $A[T]/(T^n - z)$ with n prime to the residue characteristic of A (this is the étale local description of $\rho: Y \rightarrow X$, see [FK88], I.3.2 and A.I.11). Then $\rho^{-1}D$ is the disjoint union of schemes of the form $\operatorname{Spec}(A/(z))[T]/(T^n)$ and since F is a reduced ([Gro67], 18.8.12(i)) closed subscheme of $\rho^{-1}D$ lying over D , F is the disjoint sum of copies of $D = \operatorname{Spec} A/(z)$, hence F is étale over D , as required.

We now prove (iii). Note that v_{E_0} and v_{F_0} exist by Lemma 2.1(v). Denoting $K = \operatorname{Frac} R$, and by $K(F)$, $K(E)$, $k(F_0)$, $k(E_0)$ the function fields of F , E , F_0 , E_0 , we have by Lemma 2.1 (vi) that

$$[K(F) : K(E)] = \frac{[K(F) : K]}{[K(E) : K]} = \frac{[k(F_0) : k]}{[k(E_0) : k]} = [k(F_0) : k(E_0)]$$

showing that the inertia degree of v_F over v_E equals that of v_{F_0} over v_{E_0} .

Next let e be the ramification degree of v_F over v_E . Choose affine open sets $U = \operatorname{Spec} A \subset X$ and $V = \operatorname{Spec} B \subset \rho^{-1}U$ and local parameters $f \in A$ and $g \in B$ for $E \cap U = \operatorname{Spec} A/(f)$ and $F \cap V = \operatorname{Spec} B/(g)$ respectively, so that $v_F(g) = v_E(f) = 1$ and $v_F(f) = e$. Shrinking V if necessary we may then write $f = g^e \cdot u$ with $u \in B^\times$.

Write $A_0 \stackrel{\text{df}}{=} A \otimes_R k$ and $B_0 \stackrel{\text{df}}{=} B \otimes_R k$ so that $\operatorname{Spec} A_0$ and $\operatorname{Spec} B_0$ are the restrictions of U and V to the special fibers. Then $f_0 \stackrel{\text{df}}{=} f \otimes 1 \in A_0$ and $g_0 \stackrel{\text{df}}{=} g \otimes 1 \in B_0$ are local parameters for the restrictions of $E \cap U$ and $F \cap V$ to the special fibers. In fact, since $A/(f)$ is flat over R , tensoring the exact sequence

$$0 \longrightarrow A \xrightarrow{f} A \longrightarrow A/(f) \longrightarrow 0$$

with k over R we obtain an exact sequence

$$0 \longrightarrow A_0 \xrightarrow{f_0} A_0 \longrightarrow (A/(f)) \otimes_R k \longrightarrow 0$$

showing that $f_0 \in A_0$ is regular and that $(A/(f)) \otimes_R k = A_0/(f_0)$, and similarly for g_0 . Hence $v_{F_0}(g_0) = v_{E_0}(f_0) = 1$. Finally, from the equation $f = g^e \cdot u$, $u \in B^\times$, we obtain $f_0 = g_0^e \cdot u_0$, where $u_0 \stackrel{\text{df}}{=} u \otimes 1 \in B_0^\times$, and thus $v_{F_0}(f_0) = e$ as desired. \square

2.3. An equivalence of categories. Let (R, \mathfrak{m}, k) be a complete noetherian local ring, X be a smooth integral curve over R , and D be a mark on X . We write $\text{Rev}_R^D(X)$ for the category whose objects are the tamely ramified covers of (X, D) and whose arrows are the X -morphisms. We have a “specialization” functor $\text{Rev}_R^D(X) \rightarrow \text{Rev}_k^{D_0}(X_0)$ taking a tamely ramified cover Y of (X, D) to the tamely ramified cover Y_0 of (X_0, D_0) , and a map $f: Y \rightarrow Z$ to its restriction $f_0 \stackrel{\text{df}}{=} f \times_{\text{Spec } R} \text{Spec } k: Y_0 \rightarrow Z_0$ to the special fibers. Observe that by Lemma 2.1 (i) and the definition of tamely ramified cover all objects in $\text{Rev}_R^D(X)$ and $\text{Rev}_k^{D_0}(X_0)$ are regular schemes.

Amazingly, this specialization functor $\text{Rev}_R^D(X) \rightarrow \text{Rev}_k^{D_0}(X_0)$ is an equivalence of categories (see [Wew99] 3.1.3 for the proof):

Theorem 2.3. *(Grothendieck) Let (R, \mathfrak{m}, k) be a complete noetherian local ring, X be a smooth integral curve over R , and D be a mark on X . Then restriction to the special fibers gives an equivalence of categories*

$$\text{Rev}_R^D(X) \xrightarrow{\sim} \text{Rev}_k^{D_0}(X_0)$$

For any scheme X and effective Cartier divisor D we write $\pi_1^t(X, D, \bar{x})$ for the *tame fundamental group of X with respect to D* with geometric base point $\bar{x}: \text{Spec } \Omega \rightarrow X - D$ (see [Wew99] 4.1.2, [Gro63] XIII.2.1.3 or [FK88] A.I.13). By definition, $\pi_1^t(X, D, \bar{x})$ classifies pointed tamely ramified covers of (X, D) , and thus we obtain the following (c.f. [Gro63] X.2.1)

Corollary 2.4. *With the notation and hypotheses of the previous theorem, let \bar{x}_0 be a geometric point of $X_0 - D_0$. Then the natural map*

$$\pi_1^t(X_0, D_0, \bar{x}_0) \xrightarrow{\sim} \pi_1^t(X, D, \bar{x}_0)$$

of tame fundamental groups is an isomorphism.

2.4. The ramification map. In what follows, all cohomology groups are étale cohomology groups. For a ring R and an étale sheaf F on $\text{Spec } R$ we write $H^a(R, F)$ instead of $H^a(\text{Spec } R, F)$. In particular, for a field K , $H^a(K, F)$ agrees with the Galois cohomology group $H^a(G_K, F)$ where $G_K = \text{Gal}(K_{\text{sep}}/K)$ denotes the absolute Galois group of K and where we still write F for the corresponding G_K -module.

Let K be any field, let $v: K \rightarrow \mathbb{Z} \cup \{\infty\}$ be a discrete valuation on K , and let k be its residue field. Recall that for any integer r and any integer

n prime to the characteristic of k there is a group morphism

$$\text{ram}_v: H^a(K, \mu_n^{\otimes r}) \rightarrow H^{a-1}(k, \mu_n^{\otimes(r-1)})$$

called the *residue* or *ramification map* (see [GMS03] II.7.9 or [GS06] VI.8). The ramification map has the following functorial behavior: if L is a finite extension of K and $w: L \rightarrow \mathbb{Z} \cup \{\infty\}$ is a discrete valuation with residue field l such that w extends v then we have a commutative diagram

$$\begin{array}{ccc} H^a(L, \mu_n^{\otimes r}) & \xrightarrow{\text{ram}_w} & H^{a-1}(l, \mu_n^{\otimes(r-1)}) \\ \uparrow \text{res} & & \uparrow e_{w/v} \cdot \text{res} \\ H^a(K, \mu_n^{\otimes r}) & \xrightarrow{\text{ram}_v} & H^{a-1}(k, \mu_n^{\otimes(r-1)}) \end{array}$$

where $e_{w/v}$ denotes the ramification degree of w over v .

If X is a normal integral scheme and $D \subset X$ is an irreducible Weil divisor then we write

$$\text{ram}_D: H^a(K(X), \mu_n^{\otimes r}) \rightarrow H^{a-1}(K(D), \mu_n^{\otimes(r-1)})$$

for the ramification map with respect to the discrete valuation v_D .

Lemma 2.5. *Let X be a smooth curve over a complete regular local noetherian ring, and let n be an invertible integer on X (i.e., n is prime to all residue characteristics on X). Let D be a mark on X , $U = X - D$, and denote by $j: U \hookrightarrow X$ and $i: D \hookrightarrow X$ the corresponding open and closed immersions. We have an exact Gysin sequence*

$$\begin{aligned} 0 &\rightarrow H^1(X, \mu_n^{\otimes r}) \rightarrow H^1(U, \mu_n^{\otimes r}) \rightarrow H^0(D, \mu_n^{\otimes(r-1)}) \\ &\rightarrow H^2(X, \mu_n^{\otimes r}) \rightarrow H^2(U, \mu_n^{\otimes r}) \rightarrow H^1(D, \mu_n^{\otimes(r-1)}) \\ &\rightarrow H^3(X, \mu_n^{\otimes r}) \rightarrow H^3(U, \mu_n^{\otimes r}) \rightarrow H^2(D, \mu_n^{\otimes(r-1)}) \rightarrow \dots \end{aligned}$$

where $H^a(X, \mu_n^{\otimes r}) \rightarrow H^a(U, \mu_n^{\otimes r})$ are the natural restriction maps, and the maps $H^a(U, \mu_n^{\otimes r}) \rightarrow H^{a-1}(D, \mu_n^{\otimes(r-1)})$ are compatible with the ramification maps in the sense that the following diagram commutes:

$$\begin{array}{ccc} H^a(U, \mu_n^{\otimes r}) & \longrightarrow & H^{a-1}(D, \mu_n^{\otimes(r-1)}) \\ \downarrow & & \downarrow \\ H^a(K(X), \mu_n^{\otimes r}) & \xrightarrow{\text{ram}_D} & H^{a-1}(K(D), \mu_n^{\otimes(r-1)}) \end{array}$$

Proof. Note the conclusions make sense even if D is reducible, for in this case D is the disjoint union of its irreducible components and $K(D)$ is a direct product of the corresponding function fields. The long exact Gysin

sequence will follow once we show that

$$R^q j_* \mu_n^{\otimes r} = \begin{cases} \mu_n^{\otimes r} & \text{if } q = 0 \\ i_* \mu_n^{\otimes(r-1)} & \text{if } q = 1 \\ 0 & \text{if } q \geq 2 \end{cases}$$

For then the Leray spectral sequence

$$H^p(X, R^q j_* \mu_n^{\otimes r}) \implies H^{p+q}(U, \mu_n^{\otimes r})$$

degenerates, and as i_* is an exact functor we may substitute $H^{q-1}(D, \mu_n^{\otimes(r-1)})$ for $H^{q-1}(X, i_* \mu_n^{\otimes(r-1)})$, by the Leray spectral sequence for i_* .

Since D is a mark, (X, D) is a smooth $(\text{Spec } R)$ -pair of codimension $c = 1$, and hence by purity ([Mil80] VI.5.1) we already know that $R^q j_* \mu_n^{\otimes r} = 0$ for $q \neq 0, 1$, and that $j_* \mu_n^{\otimes r} = \mu_n^{\otimes r}$. It remains to compute $R^1 j_* \mu_n^{\otimes r}$.

By [AGV73] XIX.3.3 we know that $R^1 j_* \mu_n = i_* \mathbb{Z}/n$. For the general case, consider the cup product map

$$R^0 j_* \mu_n^{\otimes(r-1)} \otimes R^1 j_* \mu_n \xrightarrow{\cup} R^1 j_* \mu_n^{\otimes r}$$

We see this is an isomorphism by looking at stalks. Since $i^* \mu_n^{\otimes(r-1)} = \mu_n^{\otimes(r-1)}$ and $R^1 j_* \mu_n = i_* \mathbb{Z}/n$, we obtain a sequence of maps

$$R^1 j_* \mu_n^{\otimes r} \xleftarrow[\approx]{\cup} \mu_n^{\otimes(r-1)} \otimes i_* \mathbb{Z}/n \xrightarrow{\text{can}} i_* i^* \mu_n^{\otimes(r-1)} \otimes i_* \mathbb{Z}/n \xrightarrow{\cup} i_* \mu_n^{\otimes(r-1)}$$

which we see are isomorphisms, again by looking at stalks. This yields the required isomorphism $R^1 j_* \mu_n^{\otimes r} = i_* \mu_n^{\otimes(r-1)}$.

Finally, to prove the compatibility with the ramification map, we may assume that D is connected. Observe that $K(D)$ is the residue field of \mathcal{O}_{v_D} . By the naturality of the Lerray spectral sequence we have a commutative diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H^a(X, \mu_n^{\otimes r}) & \longrightarrow & H^a(U, \mu_n^{\otimes r}) & \longrightarrow & H^{a-1}(D, \mu_n^{\otimes(r-1)}) \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & H^a(\mathcal{O}_{v_D}, \mu_n^{\otimes r}) & \longrightarrow & H^a(K(X), \mu_n^{\otimes r}) & \xrightarrow{(*)} & H^{a-1}(K(D), \mu_n^{\otimes(r-1)}) \longrightarrow \cdots \end{array}$$

whose rows are Gysin sequences, and $(*)$ is known to be the ramification map with respect to the valuation v_D (see [CT95] §3.3) possibly up to sign. \square

Remark 2.6. Let K be a field. We apply the previous lemma to $X = \text{Spec } K[t]$. Since $X \rightarrow \text{Spec } K$ is acyclic ([Mil80] VI.4.20) we have that $H^a(X, \mu_n^{\otimes r}) = H^a(K, \mu_n^{\otimes r})$. Moreover, any mark D is a disjoint union of closed points, hence $H^{a-1}(D, \mu_n^{\otimes(r-1)}) = \bigoplus_{P \in D} H^{a-1}(K(P), \mu_n^{\otimes(r-1)})$. Thus the Gysin sequence for X reads

$$\cdots \rightarrow H^a(K, \mu_n^{\otimes r}) \rightarrow H^a(U, \mu_n^{\otimes r}) \rightarrow \bigoplus_{P \in D} H^{a-1}(K(P), \mu_n^{\otimes(r-1)}) \rightarrow \cdots$$

where $U = X - D$. On the other hand, since étale cohomology commutes with projective limits of schemes ([Mil80] III.1.16) and $\mathrm{Spec} K(t) = \mathrm{projlim}_D X - D$, where D runs over all marks of X , by taking limits we obtain

$$\cdots \rightarrow H^a(K, \mu_n^{\otimes r}) \rightarrow H^a(K(t), \mu_n^{\otimes r}) \rightarrow \bigoplus_{P \in X^{(1)}} H^{a-1}(K(P), \mu_n^{\otimes(r-1)}) \rightarrow \cdots$$

where $X^{(1)}$ denotes the set of closed points (i.e. points of codimension 1) of X . This is just the familiar (affine) Faddeev sequence with finite coefficients ([GS06] 6.9.3), which splits into short exact sequences

$$0 \rightarrow H^a(K, \mu_n^{\otimes r}) \rightarrow H^a(K(t), \mu_n^{\otimes r}) \rightarrow \bigoplus_{P \in X^{(1)}} H^{a-1}(K(P), \mu_n^{\otimes(r-1)}) \rightarrow 0$$

via the *coresidue maps*

$$\begin{aligned} \psi_P: H^{a-1}(K(P), \mu_n^{\otimes(r-1)}) &\rightarrow H^a(K(t), \mu_n^{\otimes r}) \\ \xi &\mapsto \mathrm{cor}_{K(P)(t)|K(t)}(\xi \cdot (t - \tau_P)) \end{aligned}$$

where τ_P denotes the image of t in $K(P)$ (so that $K(P) = K(\tau_P)$) and $(t - \tau_P)$ is the image of $t - \tau_P$ in $H^1(K(P)(t), \mu_n)$.

3. SPLITTING THE RESTRICTION MAP

3.1. Setup and conventions. Henceforth we write

- (R, \mathfrak{m}, k) = complete discrete valuation ring with finite residue field k of characteristic p , and fraction field $K = \mathrm{Frac} R$ (a local field);
- π = a uniformizer of R ;
- n = integer prime to p ;
- X = a smooth integral curve over R ;
- X_0 = the special fiber of X (a smooth integral curve over k);
- $K(X)$ = the function field of X .
- $k(X_0)$ = the function field of X_0 (a global field);
- $\hat{K}(X)$ = completion of $K(X)$ with respect to the valuation defined by the special fiber X_0 . Observe that π is also a uniformizer of $\hat{K}(X)$ and that its residue field is $k(X_0)$;
- V = a fixed set of marks on X lifting each mark (i.e closed point) of X_0 , see Lemma 2.1 (vii).

By [Liu02] VIII.3.4, the set V is in 1-1 correspondence with a subset of closed points of the generic fiber $X_\eta \stackrel{\mathrm{df}}{=} X \times_{\mathrm{Spec} R} \mathrm{Spec} K$. In what follows, we will identify these two sets and refer to the unique mark $D \in V$ (or closed point $P \in X_\eta$ whose closure equals D) lifting a closed point $P_0 \in X_0$ as the *V-lift* of P_0 . For instance, if $X = \mathbb{P}_R^1 = \mathrm{Proj} R[x, y]$ and we choose the mark defined by y to be the *V-lift* of the “infinite point” of $X_0 = \mathbb{P}_k^1 = \mathrm{Proj} k[x, y]$ defined by y , then specifying the remaining *V-lifts* amounts to choosing a monic lift in $R[t]$ for each monic irreducible polynomial in $k[t]$ (where $t = x/y$).

3.2. Splitting the restriction map. In this section we construct a map

$$s = s_{V,\pi}: \mathrm{Br}(\hat{K}(X))' \rightarrow \mathrm{Br}(K(X))'$$

splitting the restriction map

$$\mathrm{res}: \mathrm{Br}(K(X))' \rightarrow \mathrm{Br}(\hat{K}(X))'$$

Here $'$ denotes the prime-to- p part of the corresponding group. In the next section we show that this map preserves the index.

Lemma 3.1. (*Tame lifting*) *The choice of V defines, for each $a \geq 0$ and $r \in \mathbb{Z}$, a group morphism*

$$\lambda_V: H^a(k(X_0), \mu_n^{\otimes r}) \rightarrow H^a(K(X), \mu_n^{\otimes r})$$

compatible with the ramification maps: for each irreducible mark $D \in V$,

$$\begin{array}{ccc} H^a(k(X_0), \mu_n^{\otimes r}) & \xrightarrow{\lambda_V} & H^a(K(X), \mu_n^{\otimes r}) \\ \mathrm{ram}_{D_0} \downarrow & & \downarrow \mathrm{ram}_D \\ H^{a-1}(k(D_0), \mu_n^{\otimes(r-1)}) & \longrightarrow & H^{a-1}(K(D), \mu_n^{\otimes(r-1)}) \end{array}$$

commutes, where the bottom arrow is given by the composition

$$H^{a-1}(k(D_0), \mu_n^{\otimes(r-1)}) \xleftarrow[\approx]{\mathrm{can}} H^{a-1}(D, \mu_n^{\otimes(r-1)}) \xrightarrow{\mathrm{can}} H^{a-1}(K(D), \mu_n^{\otimes(r-1)}).$$

Proof. Let D be a mark with support in V , and set $U = X - D$. Consider the commutative diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H^a(X, \mu_n^{\otimes r}) & \longrightarrow & H^a(U, \mu_n^{\otimes r}) & \longrightarrow & H^{a-1}(D, \mu_n^{\otimes(r-1)}) \longrightarrow \cdots \\ & & \downarrow \approx & & \downarrow & & \downarrow \approx \\ \cdots & \longrightarrow & H^a(X_0, \mu_n^{\otimes r}) & \longrightarrow & H^a(U_0, \mu_n^{\otimes r}) & \longrightarrow & H^{a-1}(D_0, \mu_n^{\otimes(r-1)}) \longrightarrow \cdots \end{array}$$

where the rows are the exact Gysin sequences for (X, D) and (X_0, D_0) respectively (see Lemma 2.5), and the vertical arrows are the natural ones (restrictions to the fibers). Since R is henselian, the left and right arrows are isomorphisms by proper base change ([Mil80] VI.2.7), hence so is the middle one by the 5-lemma.

Now define λ_D as the composition

$$\lambda_D: H^a(U_0, \mu_n^{\otimes r}) \xleftarrow[\approx]{} H^a(U, \mu_n^{\otimes r}) \xrightarrow{\mathrm{can}} H^a(K(X), \mu_n^{\otimes r})$$

Consider the set \mathcal{V} of all marks with support in V and order them by inclusion. Since étale cohomology commutes with projective limits of schemes ([Mil80] III.1.16) and

$$\mathrm{Spec} k(X_0) = \mathrm{proj} \lim_{D \in \mathcal{V}} U_0$$

taking the direct limit of the λ_D over all $D \in \mathcal{V}$ we obtain the desired map $\lambda_V: H^a(k(X_0), \mu_n^{\otimes r}) \rightarrow H^a(K(X), \mu_n^{\otimes r})$. Since by Lemma 2.5 the Gysin sequences are compatible with ramification maps and the arrow

$$H^{a-1}(k(D_0), \mu_n^{\otimes(r-1)}) \xleftarrow[\approx]{can} H^{a-1}(D, \mu_n^{\otimes(r-1)})$$

is invertible, we see that λ_V is also compatible with ramification. \square

Remark 3.2. In case $X = \mathbb{P}_R^1$, we can give a more explicit description of the tame lifting using the Faddeev sequence (see Remark 2.6). Lifting the point at infinity as in the example of Section 3.1, the map λ_V can be defined by the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^a(k, \mu_n^{\otimes r}) & \longrightarrow & H^a(k(X_0), \mu_n^{\otimes r}) & \longrightarrow & \bigoplus_{P_0 \in X_0^{(1)}} H^{a-1}(k(P_0), \mu_n^{\otimes(r-1)}) \longrightarrow 0 \\ & & \downarrow & & \downarrow \lambda_V & & \downarrow \\ 0 & \longrightarrow & H^a(K, \mu_n^{\otimes r}) & \longrightarrow & H^a(K(X), \mu_n^{\otimes r}) & \longrightarrow & \bigoplus_{P \in X_\eta^{(1)}} H^{a-1}(K(P), \mu_n^{\otimes(r-1)}) \longrightarrow 0 \end{array}$$

where each row is the split exact Faddeev sequence of Remark 2.6. The left vertical arrow is the natural one while the right vertical arrow sends, via the natural map $H^{a-1}(k(P_0), \mu_n^{\otimes(r-1)}) \rightarrow H^{a-1}(K(P), \mu_n^{\otimes(r-1)})$, the P_0 -th component to the P -th component where P denotes the generic point of the V -lift of P_0 . Explicitly, using the splitting given by the coresidue maps ψ_{P_0} , we may write an element of $H^a(k(X_0), \mu_n^{\otimes r})$ as $\alpha_0 + \sum_{P_0} \psi_{P_0}(\xi_{P_0})$ with $\alpha_0 \in H^a(k, \mu_n^{\otimes r})$ and $\xi_{P_0} \in H^{a-1}(k(P_0), \mu_n^{\otimes(r-1)})$. Then

$$\lambda_V \left(\alpha_0 + \sum_{P_0} \psi_{P_0}(\xi_{P_0}) \right) = \alpha + \sum_P \psi_P(\xi_P)$$

where P is the closed point of X_η corresponding to the V -lift of P_0 and $\alpha \in H^a(K, \mu_n^{\otimes r})$ and $\xi_P \in H^{a-1}(K(P), \mu_n^{\otimes(r-1)})$ denote the unramified lifts of α_0 and ξ_{P_0} respectively.

Lemma 3.3. *Let $\chi_0 \in H^1(k(X_0), \mathbb{Z}/n)$, and let $D_0 \subset X_0$ be the ramification locus of χ_0 . Denote by Y_0 the cyclic tamely ramified cover of (X_0, D_0) defined by χ_0 . Let $\chi = \lambda_V(\chi_0) \in H^1(K(X), \mathbb{Z}/n)$ be as in the previous lemma. Then χ defines the tamely ramified cover Y of (X, D) lifting Y_0 in Theorem 2.3, where D is the V -lift of D_0 .*

Proof. By definition of λ_V , $\chi \in H^1(X - D, \mathbb{Z}/n) \subset H^1(K(X), \mathbb{Z}/n)$ is the unique character that restricts to $\chi_0 \in H^1(X_0 - D_0, \mathbb{Z}/n) \subset H^1(k(X_0), \mathbb{Z}/n)$. Since the groups $H^1(X - D, \mathbb{Z}/n) = \text{Hom}(\pi_1^t(X, D), \mathbb{Z}/n)$ and $H^1(X_0 - D_0, \mathbb{Z}/n) = \text{Hom}(\pi_1^t(X_0, D_0), \mathbb{Z}/n)$ classify degree n (tame) cyclic Galois covers of (X, D) and (X_0, D_0) (see [FK88] I.2.11), and the restriction map

$\text{res}: H^1(X - D, \mathbb{Z}/n) \rightarrow H^1(X_0 - D_0, \mathbb{Z}/n)$ is given by the natural map $\pi_1^t(X_0, D_0) \xrightarrow{\sim} \pi_1^t(X, D)$ induced by the functor $Y \mapsto Y_0$ (see Corollary 2.4), the cyclic Galois cover Y of (X, D) defined by χ restricts to the cyclic Galois cover Y_0 of (X_0, D_0) defined by χ_0 , and we are done. \square

Theorem 3.4. *Let X , $K(X)$, $\hat{K}(X)$ and n be as in Section 3.1. Each choice of π and V as in Section 3.1 defines, for each $a \geq 0$ and all $r \in \mathbb{Z}$, a group morphism*

$$s = s_{V,\pi}: H^a(\hat{K}(X), \mu_n^{\otimes r}) \rightarrow H^a(K(X), \mu_n^{\otimes r})$$

splitting $\text{res}: H^a(K(X), \mu_n^{\otimes r}) \rightarrow H^a(\hat{K}(X), \mu_n^{\otimes r})$, that is, such that $\text{res} \circ s$ is the identity.

Proof. Let $A = \mathcal{O}_{X, \eta_0}$ where η_0 denotes the generic point of $X_0 \subset X$. Then A is a discrete valuation ring; let \hat{A} be its completion, so that $\hat{K}(X) = \text{Frac } \hat{A}$. Observe that the residue fields of both A and \hat{A} are equal to $k(X_0)$, and that π is a uniformizer for both discrete valuation rings. We have an exact Witt sequence (see [GMS03] II.7.10 and II.7.11)

$$0 \longrightarrow H^a(k(X_0), \mu_n^{\otimes r}) \longrightarrow H^a(\hat{K}(X), \mu_n^{\otimes r}) \xrightarrow{\text{ram}_{X_0}} H^{a-1}(k(X_0), \mu_n^{\otimes(r-1)}) \longrightarrow 0$$

split by the cup product with $(\pi) \in H^1(\hat{K}(X), \mu_n)$:

$$H^{a-1}(k(X_0), \mu_n^{\otimes(r-1)}) \xrightarrow{- \cdot (\pi)} H^a(\hat{K}(X), \mu_n^{\otimes r})$$

Hence each element of $H^a(\hat{K}(X), \mu_n^{\otimes r})$ can be uniquely written as a sum $\alpha_0 + \chi_0 \cdot (\pi)$ with

$$\begin{aligned} \alpha_0 &\in H^a(k(X_0), \mu_n^{\otimes r}) = H^a(\hat{A}, \mu_n^{\otimes r}) \subset H^a(\hat{K}(X), \mu_n^{\otimes r}) \quad \text{and} \\ \chi_0 &\in H^{a-1}(k(X_0), \mu_n^{\otimes(r-1)}) = H^{a-1}(\hat{A}, \mu_n^{\otimes(r-1)}) \subset H^{a-1}(\hat{K}(X), \mu_n^{\otimes(r-1)}) \end{aligned}$$

We define

$$s(\alpha_0 + \chi_0 \cdot (\pi)) = \alpha + \chi \cdot (\pi)$$

where

$$\begin{aligned} \alpha &= \lambda_V(\alpha_0) \in H^a(K(X), \mu_n^{\otimes r}) \quad \text{and} \\ \chi &= \lambda_V(\chi_0) \in H^{a-1}(K(X), \mu_n^{\otimes(r-1)}) \end{aligned}$$

are the tame lifts given by Lemma 3.1.

In order to show that $\text{res} \circ s = \text{id}$ it is enough to prove that $\alpha|_{\hat{K}(X)} = \alpha_0$ and $\chi|_{\hat{K}(X)} = \chi_0$. But this follows from the functoriality of cohomology: for instance, for α_0 , let U_0 be an open set on which α_0 is defined (i.e., α_0 belongs to the image of $H^a(U_0, \mu_n^{\otimes r}) \rightarrow H^a(k(X_0), \mu_n^{\otimes r})$), let $D_0 = X_0 - U_0$, let D be the V -lift of D_0 , and let $U = X - D$. Observe that the generic point

of X_0 belongs to U so that the natural map $H^a(U, \mu_n^{\otimes r}) \rightarrow H^a(K(X), \mu_n^{\otimes r})$ factors through $H^a(A, \mu_n^{\otimes r})$. Consequently we have a commutative diagram

$$\begin{array}{ccccc}
 H^a(U_0, \mu_n^{\otimes r}) & \xleftarrow[\approx]{\text{res}} & H^a(U, \mu_n^{\otimes r}) & & \\
 \downarrow & & \downarrow & \searrow & \\
 H^a(k(X_0), \mu_n^{\otimes r}) & \xleftarrow{\quad} & H^a(A, \mu_n^{\otimes r}) & \rightarrow & H^a(K(X), \mu_n^{\otimes r}) \\
 & \nearrow \approx & \downarrow & & \downarrow \\
 & & H^a(\hat{A}, \mu_n^{\otimes r}) & \rightarrow & H^a(\hat{K}(X), \mu_n^{\otimes r})
 \end{array}$$

and α_0 , viewed as an element of $H^a(\hat{K}(X), \mu_n^{\otimes r})$, is obtained by following the path given by U_0 , $k(X_0)$, \hat{A} , and $\hat{K}(X)$, while $\alpha|_{\hat{K}(X)}$ can be obtained by following the path given by U_0 , U , $K(X)$, and $\hat{K}(X)$. Both paths yield the same element, so this completes the proof. \square

3.3. The index does not change. In the last section, we constructed maps $s = s_{V,\pi}: H^a(\hat{K}(X), \mu_n^{\otimes r}) \rightarrow H^a(K(X), \mu_n^{\otimes r})$ splitting the restriction. In particular, since

$$\text{Br}(K(X))' = \text{inj} \lim_{n \not\equiv 0 \pmod{p}} H^2(K(X), \mu_n)$$

and similarly for $\text{Br}(\hat{K}(X))'$, we automatically obtain a map

$$s = s_{V,\pi}: \text{Br}(\hat{K}(X))' \rightarrow \text{Br}(K(X))'$$

that also splits the restriction. In this section we show that this map preserves the index. First let us recall some facts about Brauer groups of regular schemes.

Lemma 3.5. *Let X be an integral regular scheme of dimension at most 2.*

- (i) *The Brauer group $\text{Br}(X)$ of classes of Azumaya algebras on X coincides with the cohomological Brauer group $H^2(X, \mathbb{G}_m)$.*
- (ii) *There is an exact sequence*

$$0 \longrightarrow \text{Br}(X)' \longrightarrow \text{Br}(K(X))' \xrightarrow{\oplus \text{ram}_D} \bigoplus_D H^1(K(D), \mathbb{Q}/\mathbb{Z})'$$

where D runs over all irreducible Weil (or Cartier) divisors of X .

- (iii) *If X is projective over a henselian ring (A, \mathfrak{m}, k) and the special fiber $X_0 \stackrel{\text{df}}{=} X \times_{\text{Spec } A} \text{Spec } k$ has dimension at most 1 then*

$$\text{Br}(X) = \text{Br}(X_0)$$

In particular, if X_0 is a projective smooth curve over a finite field k then both groups are trivial.

Proof. For (i), see [Mil80] IV.2.16. The injectivity of $\mathrm{Br}(X) \rightarrow \mathrm{Br}(K(X))$ in (ii) is proven in [Mil80] IV.2.6, while the exactness in the middle term follows from the purity of the Brauer group (see [AG60] 7.4 or [Mil80] IV.2.18 (b), and also [Sal08], Lemma 6.6). Finally (iii) is [Gro68] 3.1 (see also [CTOP02] 1.3 for a proof using proper base change in the prime to p case), together with the fact that for any projective smooth curve C over a finite field we have $\mathrm{Br}(C) = 0$, as follows by comparing the sequence in (ii) with the one from Class Field Theory (see [GS06] 6.5):

$$0 \longrightarrow \mathrm{Br}(K(C)) \xrightarrow{\oplus_{\mathrm{ram}_P}} \bigoplus_{P \in C^{(1)}} H^1(K(P), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\Sigma} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

(here P runs over all irreducible Weil divisors of C , namely, over all its closed points). \square

Now we are ready to show

Theorem 3.6. *The map*

$$s = s_{V,\pi}: \mathrm{Br}(\hat{K}(X))' \rightarrow \mathrm{Br}(K(X))'$$

preserves the index.

Proof. Let n be prime to p . Given an arbitrary element

$$\hat{\gamma} = \alpha_0 + (\chi_0, \pi) \in {}_n\mathrm{Br}(\hat{K}(X)) = H^2(\hat{K}(X), \mu_n),$$

where $\alpha_0 \in {}_n\mathrm{Br}(k(X_0)) = H^2(k(X_0), \mu_n)$ and $\chi_0 \in H^1(k(X_0), \mathbb{Z}/n)$, let

$$\gamma = s(\hat{\gamma}) = \alpha + (\chi, \pi) \in {}_n\mathrm{Br}(K(X)) = H^2(K(X), \mu_n)$$

where $\alpha = \lambda_V(\alpha_0) \in {}_n\mathrm{Br}(K(X)) = H^2(K(X), \mu_n)$ and $\chi = \lambda_V(\chi_0) \in H^1(K(X), \mathbb{Z}/n)$ are the tame lifts of α_0 and χ_0 .

Since $\mathrm{res} \circ s = \mathrm{id}$, we have that $\mathrm{res} \gamma = \hat{\gamma}$ and therefore $\mathrm{ind} \hat{\gamma} \mid \mathrm{ind} \gamma$. To prove that $\mathrm{ind} \gamma \mid \mathrm{ind} \hat{\gamma}$ we now construct a splitting field for γ of degree $\mathrm{ind} \hat{\gamma}$ over $K(X)$.

The character χ_0 defines a cyclic extension L of $k(X_0)$ of degree equal to the order $|\chi_0|$. Since k is perfect, the normalization Y_0 of X_0 in L is a smooth curve over k , tamely ramified over X_0 (since $|\chi_0|$ is prime to $p = \mathrm{char} k$) along some mark D_0 of X_0 (the ramification locus of χ_0). By the Nakayama-Witt index formula (see [JW90] 5.15(a)) we have that

$$\mathrm{ind} \hat{\gamma} = |\chi_0| \cdot \mathrm{ind}(\alpha_0|_{k(Y_0)})$$

But since $k(Y_0)$ is a global field, the Albert-Brauer-Hasse-Noether theorem ([NSW00], Corollary 9.2.3, p. 461, and the functoriality of Corollary 9.1.8, p. 458) tells us that $\alpha_0|_{k(Y_0)}$ is cyclic, hence there is a cyclic extension of $k(Y_0)$ of degree $\mathrm{ind}(\alpha_0|_{k(Y_0)})$ that splits $\alpha_0|_{k(Y_0)}$. Corresponding to this extension there is a cyclic cover Z_0 of Y_0 , tamely ramified along some mark E_0 of Y_0 .

Let $D \subset X$ be the V -lift of D_0 . Let $\rho: Y \rightarrow X$ be the tamely ramified cover of (X, D) lifting the tamely ramified cover $\rho_0: Y_0 \rightarrow X_0$ of (X_0, D_0) , as in Theorem 2.3. Now by Lemma 2.2 (ii) the set $(\rho^{-1}V)_{\mathrm{red}}$ defines a choice

of marks on Y lifting the closed points of Y_0 . Let E be the mark on Y that lifts E_0 and whose support belongs to $(\rho^{-1}V)_{\text{red}}$. Finally define $\sigma: Z \rightarrow Y$ to be the tamely ramified cover of (Y, E) lifting the tamely ramified cover $\sigma_0: Z_0 \rightarrow Y_0$ of (Y_0, E_0) . Since

$$\begin{aligned} [K(Z) : K(X)] &= [K(Z) : K(Y)] \cdot [K(Y) : K(X)] \\ &= [k(Z_0) : k(Y_0)] \cdot [k(Y_0) : k(X_0)] \\ &= \text{ind}(\alpha_0|_{k(Y_0)}) \cdot |\chi_0| \\ &= \text{ind } \hat{\gamma} \end{aligned}$$

it is enough to show that $K(Z)$ splits γ .

Since Z is integral and regular of dimension 2, to show that $\gamma|_{K(Z)} = 0$ it is enough to show, by Lemma 3.5, that $\gamma|_{K(Z)}$ is unramified with respect to all Weil divisors on Z . On the other hand, $K(Y)$ splits χ by Lemma 3.3, hence $\gamma|_{K(Z)} = \alpha|_{K(Z)}$ and it remains to show $\alpha|_{K(Z)}$ is unramified with respect to the Weil divisors on Z . Moreover, by the construction of λ_V in the proof of Lemma 3.1, $\alpha \in H^2(U, \mu_n)$ for some open set $U \subset X$ that is the complement of a mark with support in V . Consequently, α only ramifies along marks in V . Since $\rho \circ \sigma: Z \rightarrow X$ is finite and flat (Lemma 2.2 (i)), the image of any irreducible Weil divisor in Z is also a Weil divisor in X by [Liu02] IV.3.14. That is, it cannot “contract” to a closed point. Therefore by Lemma 2.2 (ii) it is enough to show that $\alpha|_{K(Z)}$ is unramified at all marks lying over marks in V .

Now let F be an irreducible mark on Z lying over an irreducible mark G on X whose support belongs to V . Let e be the ramification degree of v_F over v_G , which equals the ramification degree of v_{F_0} over v_{G_0} as well by Lemma 2.2 (iii). By Lemma 3.1 and the functorial behavior of ramification maps under finite extensions, we have a commutative diagram

$$\begin{array}{ccccc} {}_n\text{Br}(k(X_0)) & \xrightarrow{\lambda_V} & {}_n\text{Br}(K(X)) & \xrightarrow{\text{res}} & {}_n\text{Br}(K(Z)) \\ \downarrow \text{ram}_{G_0} & \boxed{1} & \downarrow \text{ram}_G & \boxed{2} & \downarrow \text{ram}_F \\ H^1(k(G_0), \mathbb{Z}/n) & \hookrightarrow & H^1(K(G), \mathbb{Z}/n) & \xrightarrow{e \cdot \text{res}} & H^1(K(F), \mathbb{Z}/n) \\ & & \uparrow & \boxed{3} & \uparrow \\ & & H^1(k(G_0), \mathbb{Z}/n) & \xrightarrow{e \cdot \text{res}} & H^1(k(F_0), \mathbb{Z}/n) \\ & & \uparrow \text{ram}_{G_0} & \boxed{4} & \uparrow \text{ram}_{F_0} \\ & & {}_n\text{Br}(k(X_0)) & \xrightarrow{\text{res}} & {}_n\text{Br}(k(Z_0)) \end{array}$$

Here we view $H^1(k(G_0), \mathbb{Z}/n) = H^1(G, \mathbb{Z}/n)$ as the subgroup of unramified characters of $H^1(K(G), \mathbb{Z}/n)$, and similarly $H^1(k(F_0), \mathbb{Z}/n) = H^1(F, \mathbb{Z}/n) \subset H^1(K(F), \mathbb{Z}/n)$.

If $\alpha_0 \in {}_n \text{Br}(k(X_0))$, we obtain

$$\text{ram}_F(\alpha|_{K(Z)}) = e \cdot \text{ram}_{G_0}(\alpha_0)|_{K(F)} \in H^1(K(F), \mathbb{Z}/n)$$

from squares $\boxed{1} + \boxed{2}$, and we obtain

$$\text{ram}_{F_0}(\alpha_0|_{k(Z_0)}) = e \cdot \text{ram}_{G_0}(\alpha_0)|_{k(F_0)} \in H^1(k(F_0), \mathbb{Z}/n)$$

from square $\boxed{4}$. Hence $\text{ram}_F(\alpha|_{K(Z)}) = \text{ram}_{F_0}(\alpha_0|_{k(Z_0)})$ by square $\boxed{3}$, which vanishes since $\alpha_0|_{k(Z_0)} = 0$, and we are done. \square

4. INDECOMPOSABLE AND NONCROSSED PRODUCT DIVISION ALGEBRAS.

Adopt all notation from Sections 1-3. In this section we construct indecomposable division algebras over $K(X)$ and noncrossed product algebras over $K(X)$ of prime power index for all primes q with $q \neq p$. Note that noncrossed product division algebras with index equal to period over $K(X)$ for $X = \mathbb{P}_K^1$ are already known to exist by [Bru01].

4.1. Indecomposable Division Algebras over $K(X)$. We construct indecomposable division algebras over $K(X)$ by constructing them over $\hat{K}(X)$ and using the splitting $s : \text{Br}(\hat{K}(X))' \rightarrow \text{Br}(K(X))'$ from Theorem 3.6 to lift the Brauer classes to Brauer classes over $K(X)$ whose underlying division algebras are indecomposable. The construction over $\hat{K}(X)$ follows the method in [Bru96], where indecomposable division algebras of unequal prime-power index and period are shown to exist over power series fields over number fields.

We start by stating a well known lemma on the invariants of a Brauer class of a global field after a finite extension. This lemma is helpful in computing the index reduction of the Brauer class after the finite extension.

Lemma 4.1 (see [Ser79], XIII, §3). *Let $\beta \in \text{Br}(F)$ be a Brauer class over a global field F . Let L/F be a finite Galois extension. Then for all discrete valuations w in L lying over a fixed prime v of F , $\text{inv}_w(\beta_L) = e_v f_v \text{inv}_v(\beta)$.*

We now construct indecomposable division algebras over $\hat{K}(X)$.

Proposition 4.2. *Let e and i be integers satisfying $1 \leq e \leq i \leq 2e - 1$. For any prime $q \neq \text{char } k$ there exists a Brauer class $\hat{\gamma} \in \text{Br}(\hat{K}(X))$ satisfying $(\text{ind}(\hat{\gamma}), \text{per}(\hat{\gamma})) = (q^i, q^e)$ and whose underlying division algebra is indecomposable.*

Proof. Let $1 \leq t \leq e$ so that $i = 2e - t$. To prove the proposition we produce a Brauer class $\hat{\gamma} \in \text{Br}(\hat{K}(X))$ such that $(\text{ind}(\hat{\gamma}), \text{per}(\hat{\gamma})) = (q^{2e-t}, q^e)$ and $\text{ind}(q\hat{\gamma}) = q^{2e-t-1}$. Since $\text{ind}(\hat{\gamma}) = q^{2e-t}$ and $\text{ind}(q\hat{\gamma}) = q^{2e-t-1}$, by [Sal79, Lemma 3.2] the division algebra underlying $\hat{\gamma}$ is indecomposable. Choose two closed points $x_1, x_2 \in X_0$. Let v_1 and v_2 be the discrete valuations on

$k(X_0)$ corresponding to x_1 and x_2 . Let $\alpha_0 \in \text{Br}(k(X_0))$ be the Brauer class whose invariants are

$$\begin{aligned} \text{inv}_{v_1}(\alpha_0) &= \frac{1}{q^e} \\ \text{inv}_{v_2}(\alpha_0) &= -\frac{1}{q^e} \end{aligned}$$

and at all other discrete valuations v on $k(X_0)$, $\text{ram}_v(\alpha_0) = 0$. The Brauer class α_0 exists by Hasse's residue theorem ([GS06, 6.5.4]) and the fact that $k(X_0)$ is a global field. Let $\xi_{v_i} = \text{ram}_{v_i}(\alpha_0) \in H^1(k(v_i), \mathbb{Q}/\mathbb{Z})$. Let $k(X_0)_{v_i}$ be the completion of $k(X_0)$ at the valuation v_i and choose unramified characters $\theta_{v_i} \in H^1(k(X_0)_{v_i}, \mathbb{Q}/\mathbb{Z})$ of order q^t . By the Grunwald-Wang theorem there exists a global character $\theta_0 \in H^1(k(X_0), \mathbb{Q}/\mathbb{Z})$ of order q^e with restrictions θ_{v_i} at v_i for $i = 1, 2$.

Set $\hat{\gamma} = \alpha_0 + (\theta_0, \pi) \in \text{Br}(\hat{K}(X))$, an element with period q^e . We claim that $\text{ind}(\hat{\gamma}) = q^{2e-t}$ and $\text{ind}(q\hat{\gamma}) = q^{2e-t-1}$. By the Nakayama-Witt index formula (see [JW90] 5.15(a)) we have $\text{ind} \hat{\gamma} = |\theta_0| \cdot \text{ind}(\alpha_0|_{k(X_0)(\theta_0)})$ where $\alpha_0|_{k(X_0)(\theta_0)}$ is the restriction of α_0 to $k(X_0)(\theta_0)$, the finite extension defined by the character θ_0 . By construction, $|\theta_0| = q^e$ so it is only left to show that $\text{ind}(\alpha_0|_{k(X_0)(\theta_0)}) = q^{e-t}$. Since $k(X_0)(\theta_0)$ is a finite extension of $k(X_0)$, $k(X_0)(\theta_0)$ is a global field and

$$\text{ind}(\alpha_0|_{k(X_0)(\theta_0)}) = \text{per}(\alpha_0|_{k(X_0)(\theta_0)}) = \text{lcm}_w(|\text{inv}_w(\alpha_0|_{k(X_0)(\theta_0)})|)$$

where the least common multiple is taken over all discrete valuations w of $k(X_0)(\theta_0)$. This shows, by our assumptions on α_0 , that for all discrete valuations w of $k(X_0)(\theta_0)$,

$$\text{inv}_w(\alpha_0|_{k(X_0)(\theta_0)}) = \begin{cases} 0, & \text{if } w \text{ does not lie over } v_i \text{ for } i = 1, 2 \\ \frac{\pm |(\theta_0)_{v_i}|}{q^e}, & \text{if } w \text{ lies over } v_i \text{ for } i = 1, 2 \end{cases}$$

By our assumption on θ_0 , $|(\theta_0)_{v_i}| = q^t$ for $i = 1, 2$ and therefore we have $\text{ind}(\alpha_0|_{k(X_0)(\theta_0)}) = q^{e-t}$ and $\text{ind}(\hat{\gamma}) = q^{2e-t}$.

A similar calculation for $q\hat{\gamma}$ gives $|q\theta_0| = q^{e-1}$ and $\text{ind}(q\alpha_0|_{k(X_0)(q\theta_0)}) = q^{e-t}$ since by the same reasoning,

$$\text{inv}_w(q\alpha_0|_{k(X_0)(q\theta_0)}) = \begin{cases} 0, & \text{if } w \text{ does not lie over } v_i \text{ for } i = 1, 2 \\ \frac{\pm |(q\theta_0)_{v_i}|}{q^{e-1}}, & \text{if } w \text{ lies over } v_i \text{ for } i = 1, 2 \end{cases}$$

and $|(q\theta_0)_{v_i}| = q^{t-1}$ for $i = 1, 2$. We conclude $\text{ind}(q\hat{\gamma}) = q^{2e-t-1}$. \square

Theorem 4.3. *Let k , X_0 , K and X be as in Section 3.1 and let q be a prime with $q \neq \text{char } k$. Fix integers e and i satisfying $1 \leq e \leq i \leq 2e - 1$. Then there exists an indecomposable division algebra D over $K(X)$ satisfying $(\text{ind}(D), \text{per}(D)) = (q^i, q^e)$.*

Proof. Choose e and i so that $1 \leq e \leq i \leq 2e - 1$. By Proposition 4.2 there exists a Brauer class $\hat{\gamma} \in \text{Br}(\hat{K}(X))$ satisfying $(\text{ind}(\hat{\gamma}), \text{per}(\hat{\gamma})) = (q^i, q^e)$ and

whose underlying division algebra is indecomposable. By Theorem 3.6, $\gamma = s(\hat{\gamma}) \in \text{Br}(K(X))$ has index q^i . Since s is a splitting of the restriction map, we also have $\text{per}(\gamma) = q^e$. To finish the proof we show the division algebra underlying γ is indecomposable. If $\gamma = \beta_1 + \beta_2$ with $\text{ind}(\beta_1)\text{ind}(\beta_2) = \text{ind}(\gamma)$ represents a nontrivial decomposition of the division algebra underlying γ , then $\hat{\gamma} = \text{res}_{\hat{K}(X)}(\beta_1) + \text{res}_{\hat{K}(X)}(\beta_2)$. Since the index can only decrease under $\text{res}_{\hat{K}(X)}$ we have $\text{ind}(\hat{\gamma}) = \text{ind}(\text{res}_{\hat{K}(X)}(\beta_1))\text{ind}(\text{res}_{\hat{K}(X)}(\beta_2))$. This represents a nontrivial decomposition of the division algebra underlying $\hat{\gamma}$, a contradiction. \square

Remark 4.4. In the case $X = \mathbb{P}_R^1$, it is not hard to construct $\hat{\gamma}$ which satisfies the conclusions of Proposition 4.2 and can be seen to have $\text{ind}(\hat{\gamma}) = \text{ind}(s(\hat{\gamma}))$ without the use of Theorem 3.6. Choose e, i, t so that $1 \leq e \leq i \leq 2e - 1$ and $i = 2e - t$. Then, as in the proof of Proposition 4.2, choose a single closed point x_0 in $X_0 = \mathbb{P}_k^1$ of degree q^{e-t} . Let $\xi \in H^1(k, \mathbb{Z}/n)$ be a character of order q^{2e-t} where n is an integer prime to p with $q^i \mid n$. Set $\alpha_0 = (\xi, \pi_{x_0})$ where π_{x_0} is the irreducible polynomial corresponding to the closed point x_0 . Then,

$$\text{ram}_x(\alpha_0) = \begin{cases} 0, & \text{if } x \neq x_0 \text{ and } x \neq \text{the point at infinity} \\ \text{res}_{k|k(x_0)} \xi, & \text{if } x = x_0 \end{cases}$$

Set $\theta_0 = q^{e-t}\xi \in H^1(k, \mathbb{Z}/n) \hookrightarrow H^1(k(t), \mathbb{Z}/n)$. Set $\hat{\gamma} = \alpha_0 + (\theta_0, p)$. Since $\text{per}(\alpha_0) = |\text{inv}_{x_0} \alpha_0| = q^e$ and $\text{per}((\theta_0, p)) = q^e$, $\text{per}(\hat{\gamma}) = q^e$. Using the same strategy as Proposition 4.2 shows that $\text{ind}(\hat{\gamma}) = q^{2e-t}$ and $\text{ind}(q\hat{\gamma}) = q^{2e-t-1}$. Therefore, $\hat{\gamma}$ satisfies the conclusions of Proposition 4.2. We now check $\text{ind}(s(\hat{\gamma})) = q^{2e-t}$. Let $\theta = s(\theta_0)$ which is the unique lift of the constant extension θ_0 to $H^1(K(t), \mathbb{Z}/n)$. The character θ defines a p -unramified extension $L/K(t)$ of degree q^e . Then, $s(\hat{\gamma})_L = (s(\xi), s((\pi_{x_0})))_L + (\theta, p)_L = (s(\xi), s((\pi_{x_0})))_L$. Thus $\text{ind}(s(\hat{\gamma})_L) = \text{ind}((s(\xi), s((\pi_{x_0})))_L) \leq |\xi|/|\theta| = q^{e-t}$ since L is contained in the p -unramified constant extension defined by $s(\xi)$ which is a lift of ξ . Therefore, $\text{ind}(s(\hat{\gamma})) \leq [L : K(t)]q^{e-t} = q^{2e-t} = \text{ind}(\hat{\gamma})$. Since $\text{ind}(s(\hat{\gamma})) \geq \text{ind}(\hat{\gamma})$, we get the equality $\text{ind}(s(\hat{\gamma})) = \text{ind}(\hat{\gamma})$.

Remark 4.5. Set $R = \mathbb{Z}_p$ and $K = \mathbb{Q}_p$ and let X be as in 3.1. By [Sal98] the index of any Brauer class in $\text{Br}(K(X))$ divides the square of its period. Let q be a prime with $q \neq p$. Theorem 4.3 shows that over $K(X)$ there exist indecomposable division algebras of index-period combination (q^i, q^e) for all $1 \leq e \leq i \leq 2e - 1$ and all primes $q \neq p$. In [Sur08], Suresh builds on the work of [Sal07] to show that if $L/\mathbb{Q}_p(t)$ is a finite extension containing the q -th roots of unity, then every element in $H^2(L, \mu_q)$ is a sum of at most two symbols. In particular, a division algebra over L of index q^2 and period q must be decomposable as it is the sum of two symbols each of index q .

4.2. Noncrossed products over $K(X)$. In this section we construct noncrossed product division algebras over $K(X)$. Throughout this section we adopt all notation from Section 3.1. In particular, K is the fraction field of

R , a complete discrete valuation ring with uniformizer π and residue field k , a field of characteristic p and X is a smooth curve over R . We use the same strategy as in Section 4.1, that is, we construct noncrossed products of q -power index (q a prime, $q \neq p = \text{char } k$) over $\hat{K}(X)$ and use the splitting $s : \text{Br}(\hat{K}(X))' \rightarrow \text{Br}(K(X))'$ from Theorem 3.6 to lift the noncrossed products to $K(X)$.

The method of constructing the noncrossed products over $\hat{K}(X)$ follows the method in [Bru95] where noncrossed products over $\mathbb{Q}(t)$ and $\mathbb{Q}((t))$ are constructed. In order to mimic the construction in [Bru95] we need only note that both the Čebotarev density theorem, and the Grunwald-Wang theorem hold for global fields which are characteristic p function fields. After noting these two facts, the reader can check that the arguments in [Bru95] apply directly to obtain noncrossed products over $\hat{K}(X)$ of index and period given below.

Index and Period Setup 4.6. Let K , R , k , X and X_0 be as in Section 3.1. For any positive integer a , let ϵ_a denote a primitive a -th root of unity. Set r and s to be the maximum integers such that $\mu_{q^r} \subset k(X_0)^\times$ and $\mu_{q^s} \subset k(X_0)(\epsilon_{q^{r+1}})^\times$. Let n and m be integers such that $n \geq 1$, $n \geq m$, and $n, m \in \{r\} \cup [s, \infty)$. Let a and l be integers such that $l \geq n + m + 1$ and $0 \leq a \leq l - n$. See [Bru95, p.384-385] for more information regarding these constraints.

Theorem 4.7. *Let K , R , k , X and X_0 be as in Section 3.1. Let q be a prime, $q \neq p = \text{char } k$ and let a and l be integers satisfying the properties of 4.6. Then there exists noncrossed product division algebras over $\hat{K}(X)$ of index q^{l+a} and period q^l .*

Corollary 4.8. *Let K , R , k , X , X_0 , q , a and l be as Theorem 4.7. Then, there exists noncrossed product division algebras over $K(X)$ of index q^{l+a} and period q^l .*

Proof. Let \hat{D} be a noncrossed product over $\hat{K}(X)$ of index q^{l+a} , period q^l . Let D be the division algebra in the class of $s([\hat{D}]) \in \text{Br}(K(X))$. By Theorem 3.6 we know that $\text{ind}(D) = \text{ind}(\hat{D})$. Assume by way of contradiction that D is a crossed product with maximal Galois subfield $M/K(X)$. Then $M\hat{K}(X)$ splits \hat{D} , is of degree $\text{ind}(\hat{D})$ and is Galois. This contradicts the fact that \hat{D} is a noncrossed product. \square

Remark 4.9. Noncrossed products were already known to exist over $\mathbb{Q}_p(t)$ by [Bru01]. In the noncrossed products of [Bru01] the index always equals the period. This is not the case in the above construction.

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